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Wormhole Solutions in Superstring Theory

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The wormhole is discussed in 4-dimensional superstring theory; the corresponding wormhole equation is deduced, from which both the analytical and numerical solutions are given with three different cases of cosmological constant.

A wormhole is a Euclidean field configuration in some field theory containing gravity, consisting of two asymptotically flat regions connected by a tube, or throat. Wormholes were introduced in the 1950s by Wheeler [1]. A specific example is the Schwarzschild bridge, which is a slice through a black hole joining two asymptotic regions. By the end of the 1980s, topological structures of space-time were being widely discussed in relation to the study of quantum cosmology. Such concepts as 4-dimensional wormholes, baby universes, and their effects on spacetime coupling constants have provoked great interest among theoretical physicists [2–8]. Since this kind of wormhole can join spaces with different topologies, it represents tiny quantum fluctuations of space. From the mathematical point of view, the condition of a wormhole existing on a 4-dimensional asymptotic flat manifold M_4 is that the Ricci tensor of M_4 has negative eigenvalues somewhere on M_4 [9]. In the pure gravitation case, Hawking discussed wormholes in which two baby universes are connected [2]. The wormhole with a stable Euclidean acrion was first discovered by Giddings and Strominger [10] in a theory with a spontaneously broken Abelian internal symmetry—the theory of a Goldstone boson, or axion, minimally coupled to Einstein gravity. From the point of view of physics, such wormhole solutions can also be used to study the

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dimensions of spacetime, quantum chaos [11–13], and the third quantization problem of baby universes [8, 14–17]. Various wormhole solutions have been discovered, e.g., wormholes in complex scalar fields both with [18] and without [19] broken symmetry, wormholes in Yang–Mills gauge field [20–22], Skyrme field [23], higher dimensional space-time [24–26], scalar field [27], spinor field [28], and superstring theory [29]. Wormhole solutions with topology $S^1 \otimes S^2$ are discussed in ref. 30, and wormholes in scalar-tensor gravitation theory are discussed in ref. 31. In refs. 32–35 the Wheeler–de Witt equation is used to discuss the wormhole wavefunction. The Hilbert space structures of wormholes are discussed in refs. 36–38.

In this paper, wormholes are studied in a 4-dimensional superstring theory; the corresponding wormhole equations are deduced and solved, from which both analytical and numerical solutions are given with three different cases of cosmological constant.

Since the superstring model can be regarded as a general quantum gravity theory, it is natural to consider the importance of the cosmological constant in it. In general, a string field equation can be deduced from the conformal σ model; an effective action of a 4-dimensional superstring containing the cosmological constant can be expressed as [10]

$$S = \int d^4x \sqrt{g} e^{\beta\varphi} \left[-R - (\nabla\varphi)^2 + H^2 + \frac{\alpha'}{8} (R^2 - F^2) + 2\Lambda \right] \quad (1)$$

where α' is the coefficient of string tension, φ is a scalar vacuum field, and H is the modified Kalb–Ramond field intensity, which can be expressed as a Yang–Mills–Chern–Simons 3-form ω_{8_r} and a Chern–Simons 3-form ω_{3L}, that is,

$$H = dB - \omega_{3r} + \omega_{3L}$$
$$\omega_{3r} = \mathrm{tr}(AF - \tfrac{1}{3}A^3)$$
$$\omega_{3L} = \mathrm{tr}(\omega R - \tfrac{1}{3}\omega^3)$$

The spherically symmetric metric is adopted for a 4-dimensional superstring model

$$dS^2 = dt^2 + a^2(t)\, d\Omega_3^2 \quad (2)$$

Here $d\Omega_3^2$ is the metric of three-dimensional sphere. The axion field intensity takes the form

$$H = h(t)\varepsilon \quad (3)$$

with ε the volume element of S^3. One has

$$\int_{S^3} \varepsilon = 2\pi^2 a^3(t) \tag{4}$$

From the conservation equation of the axion field $dH^* = dH = 0$ one can deduce

$$h(t) = \frac{n}{f^2 a^3} \qquad (n \in Z) \tag{5}$$

Here $*$ denotes Hodge couple. f is a parameter related to the string model. $dH = 0$ is equivalent to the cancellation condition of gauge anomaly and gravity anomaly, $\mathrm{tr}\, R^2 = \mathrm{tr}\, F^2$, and thus one has the quantized axion field on S^3

$$\int_{S^3} H = \frac{2\pi n^2}{f^2} \tag{6}$$

provided that the scalar vacuum field φ is independent of time, i.e.,

$$\frac{d\varphi}{dt} = 0 \tag{7}$$

and from Eqs. (5) and (7), one obtains the Einstein field equation for scale factor $a(t)$,

$$3\left(\frac{\dot{a}^2}{a^2} - \frac{1}{a^2}\right) = -3e^{\beta\varphi}\,\frac{n^2}{f^2 a^6} - 2\Lambda \tag{8}$$

which can be rewritten as

$$\dot{a}^2 = 1 - \frac{r_n^2}{a^4} - \frac{2}{3}\,\Lambda a^2 \tag{9}$$

Here $r_n^2 = e^{\beta\varphi} n^2 / f^2$ is the parameter of the superstring wormhole.

Now let us solve Eq. (9) under three different cases of Λ.

1. $\Lambda < 0$

From Eq. (9), when $\dot{a} = 0$ one has $\ddot{a} > 0$, which means there exists a minimum of a, which satisfies the relationship

$$a_{\min} < \sqrt[4]{\frac{e^{\beta\varphi} n^2}{f^2}} \tag{10}$$

In the limit when $t \to \infty$, one obtains

$$a_{\min} = \sqrt{\frac{3}{2|\Lambda|}} \tag{11}$$

One thus has the asymptotic solution of Eq. (9)

$$a(t) = \sqrt{\frac{3}{2|\Lambda|}} \sinh \frac{t}{\sqrt{3/(2|\Lambda|)}} \tag{12}$$

Therefore, one obtains a wormhole solution which is a generalized De Sabbata–Sivaram wormhole.

2. $\Lambda = 0$

Equation (9) becomes

$$\dot{a}^2 = 1 - \frac{r_n^2}{a^4} \tag{13}$$

the solution of which is

$$\frac{t}{\sqrt{r_n}} = \frac{1}{\sqrt{2}} F\left[\arccos \frac{\sqrt{r_n}}{a}, \frac{1}{\sqrt{2}}\right] - \sqrt{2} E\left[\arccos \frac{\sqrt{r_n}}{a}, \frac{1}{\sqrt{2}}\right] + \frac{1}{\sqrt{r_n} a} \sqrt{a^4 - r_n^2} \tag{14}$$

where F and E are elliptic integrals of the first and the second kinds, respectively. This is just the Giddings–Strominger wormhole. The corresponding $a_{\min}$ is

$$a_{\min} = \sqrt[4]{\frac{e^{\beta\varphi} n^2}{f^2}} \tag{15}$$

In the limit when $t \to \infty$, one has the asymptotic solution

$$a(t) = t \tag{16}$$

3. $\Lambda > 0$

In this case, there exists a critical value of Λ; when $\Lambda > \Lambda_{\text{crit}}$, the requirement for existence of $a_{\min}$ cannot be satisfied. Therefore, the wormhole solution does not exist. The Λ_{crit} can be deduced from Eq. (9) as

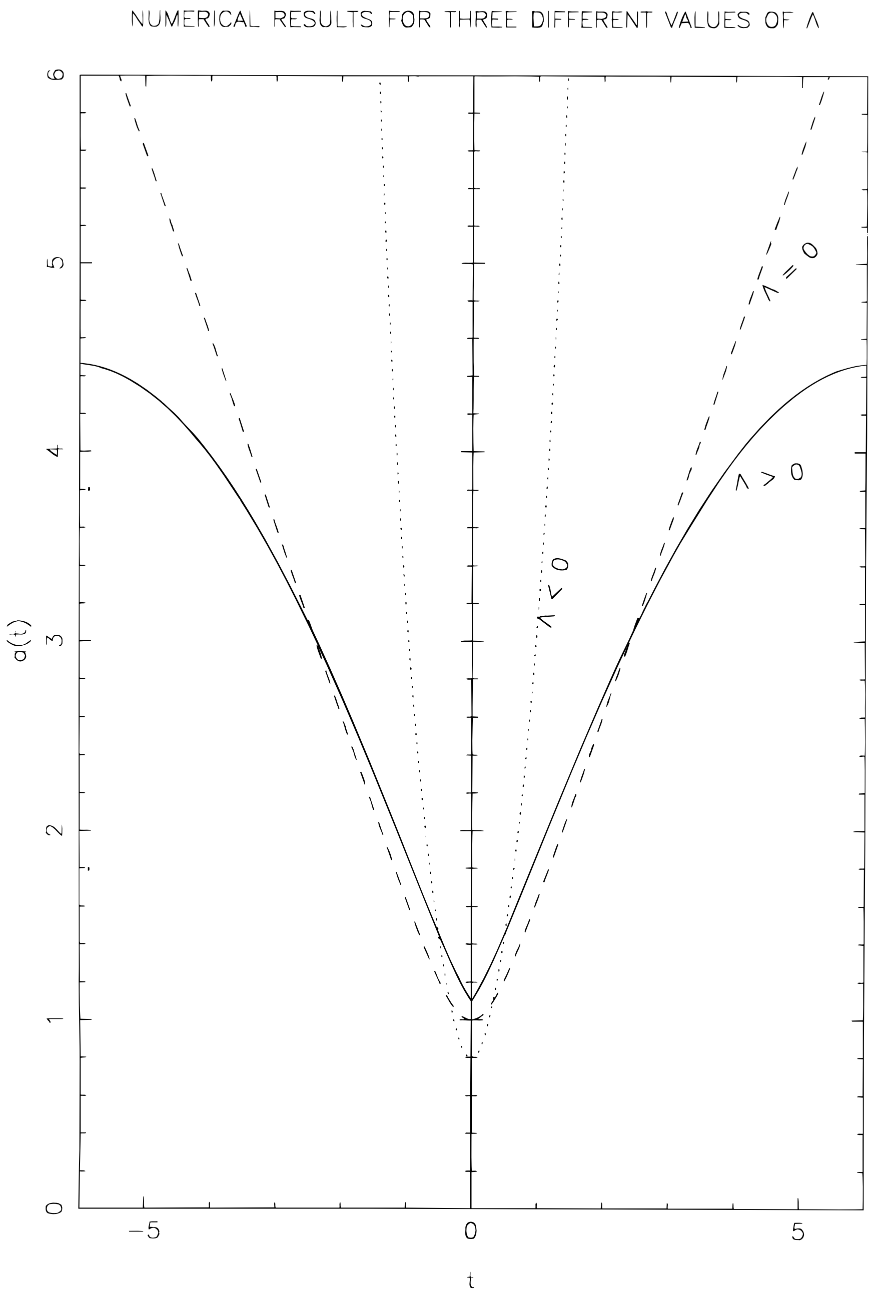

Fig. 1. Numerical results for the cosmological scale factor $a(t)$ versus time t calculated from Eq. (9). The unit of $a(t)$ in the plot is assumed to be r_n^2, i.e., $r_n^2 = e^{\beta\varphi}n^2/f^2 = 1$. Curves corresponding to three different values of Λ are indicated.

$$\Lambda_{\text{crit}} = \frac{1}{\sqrt{3}(e^{\beta\varphi}n^2/f^2)^{1/2}} \tag{17}$$

When $0 < \Lambda < \Lambda_{\text{crit}}$, from Eq. (9) one has $\ddot{a} > 0$, so $a_{\min}$ exists. One thus has as the solution of Eq. (9)

$$t = \frac{\beta}{(2\Lambda/3)^{1/2}[\alpha(\beta - \gamma)]^{1/2}}\, \Pi\left[h\left(\frac{a^2}{r_n^2}\right), \frac{\alpha - \beta}{\alpha}, \left[\frac{\gamma(\beta - \alpha)}{\alpha(\beta - \gamma)}\right]^{1/2}\right] \tag{18}$$

where

$$h(\chi) = \arcsin\left[\frac{\alpha(\chi - \beta)}{\chi(\alpha - \beta)}\right]^{1/2} \tag{19}$$

Π is the elliptic integral of the third kind, while α, β, γ ($\alpha > \beta > 0 > \gamma$) are three real roots of the equation

$$x^3 - \frac{1}{(2\Lambda/3)r_n^2}\, x^2 + \frac{1}{(2\Lambda/3)r_n^2} = 0 \tag{20}$$

In this case

$$a_{\min} > \sqrt[4]{\frac{e^{\beta\varphi}n^2}{f^2}} \tag{21}$$

and there exists an $a_{\max}$ of the radius of the wormhole throat.

Because there is a limit to the length of wormhole throat, the central singularity of the wormhole solutions is avoided. The Pontryagin topological number can be expressed as

$$d\omega_{cs} = \frac{1}{8\pi^2}\, \text{tr}\, F^2 \tag{22}$$

We give in Fig. 1 the numerical results for the cosmological scale factor versus time t. The main features of the analytical solutions corresponding to the three cases of the cosmological constant Λ are confirmed in the plot.

REFERENCES

1. J. A. Wheeler, *Phys. Rev.* **97** (1955), 511.
2. S. W. Hawking, *Phys. Lett. B* **195** (1987), 337.
3. A. Linde, *Phys. Lett. B* **200** (1988), 272.
4. S. Coleman, *Nucl. Phys. B* **307** (1988), 864.
5. S. Coleman, Preprint, HOTP-88/A022.
6. I. Klebanov, L. Susskind, and T. Banks, *Nucl. Phys. B* **317** (1989), 665.
7. S. B. Giddings and A. Strominger, *Nucl. Phys. B* **307** (1988), 854.

8. S. B. Giddings and A. Strominger, *Nucl. Phys. B* **321** (1989), 481.
9. J. Gheegen and D. Grommol, *Ann. Math.* **96** (1972), 413.
10. S. B. Giddings and A. Strominger, *Nucl. Phys. B* **306** (1988), 890.
11. I. V. Volovich, *Phys. Lett. B* **219** (1989), 66.
12. A. Stroimnger, *Phys. Rev. Lett.* **52** (1984), 1733.
13. G. V. Larrelashvili, V. A. Rubakov, and P. G. Tinyakov, *Mod. Phys. Lett. A* **13** (1988), 1213.
14. A. Strominger, *Phil. Trans. R. Soc. Lond. A* **329** (1989), 395.
15. A. Hosoya and M. Morikawa, *Phys. Rev. D* **39** (1989), 1123.
16. M. T. Duncan, Preprint, UMN-TH-716/90.
17. Y. Peley, *Class. Quantum Grav.* **8** (1991), 827.
18. K. Lee, *Phys. Rev. Lett.* **61** (1988), 263.
19. S. Coleman and K. Lee, *Nucl. Phys. B* **329** (1990), 387.
20. A. Hosoya and W. Ogura, *Phys. Lett. B* **225** (1989), 17.
21. A. K. Guta, J. Hughes, J. Preskill, and M. B. Wise, *Nucl. Phys. B* **333** (1990), 160.
22. S. J. Rey, *Nucl. Phys. B* **336** (1990), 146.
23. A. Iwazaki, *Phys Rev. D* **41** (1990), 3280.
24. R. C. Myers, *Phys. Rev. D* **38** (1988), 1327.
25. R. C. Myers, *Nucl. Phys. B* **323** (1989), 225.
26. Y. G. Shen, H. Y. Gao, Z. Q. Tan, and H. G. Ding, *Phys. Rev. D* **44** (1991), 1330.
27. C. P. Bergess and S. Hagan, *Phys. Rev. D* **43** (1991), 1869.
28. K. Lee and S. M. Smirnakis, Preprint, HUTP-89/A022.
29. S. B. Giddings and A. Strominger, *Phys. Lett. B* **250** (1989), 46.
30. S. A. Ridgway, *Phys. Rev. D* **50** (1994), 892.
31. F. S. Accetta, A. Chodos, and B. Shao, *Nucl. Phys. B* **333** (1990), 221.
32. S. W. Hawking, *Phys. Rev. D* **37** (1988), 904.
33. S. W. Hawking and D. N. Page, *Phys. Rev. D* **42** (1990), 2738.
34. L. M. Campbell and L. J. Garny, *Phys. Lett. B* **254** (1991), 49.
35. U. Bleyer, V. D. Ivashchuk, V. N. Melnikov, and A. Zhuk, *Nucl. Phys. B* **429** (1994), 177.
36. L. J. Gray, *Phys. Rev. D* **48** (1993), 1710.
37. G. A. Mena Marugan, *Phys. Rev. D* **50** (1994), 3923.
38. C. Barcelo, L. J. Garay, P. F. Gonzalez-Diaz, and G. A. Mena Marnga, *Phys. Rev. D* **53** (1996) 3162.